Reliable Sliding Mode Control of Fast Sampling Singularly Perturbed Systems: A Redundant Channel Transmission Protocol Approach

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***Abstract*—This paper endeavors to investigate the output-feedback sliding mode control (SMC) issue of the networked singularly perturbed systems (SPSs) under fast sampling. In order to improve the reliability of the network communication, a redundant channel transmission protocol is introduced in the SMC design. Based on the measurement outputs, a sliding function is constructed with the consideration of the transmission protocol. With the aid of some appropriate Lyapunov functions, the sufficient conditions are derived to ensure the mean-square exponentially ultimately boundedness of the sliding mode dynamics and the reachability of the specified sliding surface. Moreover, a convex optimization algorithm is formulated to solve the output-feedback SMC law via searching the available upper bound of the singularly perturbed parameter. Finally, an operational amplifier circuit is exploited to explore the influences from the redundant channel transmission protocol to the output-feedback SMC performance and the estimated ε-bound.**

***Index Terms*—Reliability; Redundant Channel Transmission Protocol; Sliding Mode Control; Singularly Perturbed Systems.**

I. INTRODUCTION

Singularly perturbed system (SPS) has been widely employed to model the dynamic system with multiple time-scale phenomena, which frequently occur in, for example, the electrical circuits [2], engineering biomolecular systems [24], and advanced heavy water reactor [22]. In a SPS, a key feature is that the degree of separation between the "slow" and "fast" modes of the systems are multiplied via a small positive parameter, i.e., the singular perturbed parameter ε. Over the past few decades, a considerable research attention has been focused on the stability analysis and the control/filter synthesis issues for the SPSs, see [4], [34] for continuous-time case and [8], [18], [31] for discrete-time case.

As an effective robust control scheme in engineering applications, sliding mode control (SMC) approach has been widely employed to tackle the *matched* parameter variations and external disturbances [5], [11], [12], [17], [27], [28], [36]. To date, some particular research interests have been paid to the SMC synthesis of the *continuous-time* SPSs. For example, by constructing a proper integral sliding surface, the passivity-based SMC issue of uncertain SPSs was addressed in [9] and [33]. With the aid of a singular perturbation approach, the authors in [10] studied the SMC problem of the uncertain dynamical systems with the time varying input delay. However, compared with the well-studied SMC problems of *continuous-time* SPSs, the research on the SMC problem of *discrete-time* SPSs is still on its early stage despite the fact that the *discrete-time* SPSs are important for the network-based communication or computer simulation. It is noted that according to the different sampling rates, there are two representative discrete-time SPS models, one is the fast-sampling SPSs [8], [18] and another is the slow-sampling SPSs [31]. Therefore, the main challenging to deal with the SMC problem of the *discrete-time* SPSs could be how to design a proper SMC law by taking the specific singular perturbed structure of discrete-time fast/slow-sampling SPSs into account under the purpose of avoiding the possible ill-conditioned numerical problems. These facts constitute one part of the motivations for the present research.

With rapid advances in communication network nowadays, the networked control systems (NCSs) that facilitate the remote execution of certain tasks such as monitoring and control have attracted considerable research attentions over the past few decades. The NCSs have many merits such as low cost, reduced weight and power, and easy manipulation. Nevertheless, the implementation of communication networks between system components (e.g., sensors, controller and actuators) has largely raised the level of complexities in the analysis and synthesis of the overall NCSs mainly due to the inherent limited bandwidth [14], [29], [32], [42]. It is well recognized that if not properly handled, the network-induced phenomena could cause performance degradation of the overall NCSs. Among various network-induced phenomena, the packet dropout has gained much research attenuation, because it widely exists in many communication networks [26]. Specifically, by introducing a compensation strategy to the packet dropout, Niu & Ho in [23] addressed the SMC design problem subject to packet losses. Following this excellent work, some interesting results have been reported in the literature concerning the SMC problems subject to missing measurements, see [3], [19], [25], [40] and the references therein. In fact, these literatures dealt with the packet dropouts via a *passive* way, that is, to compensate the packet dropouts properly in designing SMC.

As a *proactive* way to face packet dropouts, the redundant channel transmission protocol (RCTP) has been widely utilized in IEC 62439-3-based industrial Ethernet [7] and some

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industrial systems and critical infrastructures such as power systems [39]. The key idea in RCTP is that if the primary channel suffers certain communication failure, which can be detected by means of software or hardware devices, other channels (i.e., redundant channels) will be automatically activated to protect the key data and thus improve the reliability of the communication network in a great sense. In the recent years, some particular research interests have been focused on the control/filtering under RCTP, see [21], [30], [38], [41] for example. Unfortunately, till now, the problem of SMC design subject to packet dropouts by using RCTP has not received proper research attention, not to mention the fast-sampling SPSs become another research focus.

Motivated by the above discussions, in this paper, we endeavor to solve the output-feedback SMC problem for the fast-sampling SPSs. In order to improve the communication reliability, the RCTP is applied between the sensors and the controller. This appears to be a nontrivial task mainly due to the following three essential difficulties: 1) only with the measurement outputs, how to design a suitable sliding function by considering the structure characteristics of RCTP? 2) how to deal with the redundant channels in analyzing the stability of the sliding mode dynamics and the reachability of the sliding surface? and 3) how to estimate ε-bound for the fast-sampling SPSs in an easy-to-implement method? It is, therefore, the purpose of this paper to shorten such a gap by launching a systematic investigation.

The main contributions of this paper are highlighted as follows:

1) To cope with the impact of RCTP, a measured output-based sliding function is developed for the fast-sampling SPSs.
2) By utilizing some stochastic analysis techniques, sufficient conditions for ensuring the stability of the sliding mode dynamics and the reachability of the specified sliding surface are established.
3) A convex optimization problem is formulated to design the SMC strategy with estimating the upper bound of the singular perturbation parameter.
4) The impacts from the RCTP to the SMC performance and the estimation of ε-bound are explored via an operational amplifier circuit.

Notation: Throughout this paper, the following mathematical notations are used:

- $\mathbb{R}^{n\times m}$: The set of all $n\times m$ real matrices.
- $X\geq Y$ (X and Y are symmetric matrices): $X-Y$ is positive semi-definite matrix.
- $\mathbf{E}\{x\}$: The expectation of the stochastic variable x.
- $\mathrm{diag}\{\cdot\}$: A block diagonal matrix.
- "$\star$": An ellipsis for terms induced for symmetry in symmetric block matrices.
- $\mathrm{sgn}(s) = \begin{bmatrix}\mathrm{sgn}(s_1) & \mathrm{sgn}(s_2) & \cdots & \mathrm{sgn}(s_m)\end{bmatrix}^{\mathrm{T}}$ with $\mathrm{sgn}(s_i)=\begin{cases}1, & s_i>0,\\ 0, & s_i=0, \quad \text{for } i=1,2,\ldots,m.\\ -1, & s_i<0,\end{cases}$

II. Problem Formulation

A. Fast-sampling SPSs

Consider the following fast-sampling discrete-time SPS:

$$x(k+1)=A_\varepsilon x(k)+B_\varepsilon\left(u(k)+w(k)\right), \tag{1}$$

with $x(k)\triangleq\begin{bmatrix}x_1^{\mathrm{T}}(k) & x_2^{\mathrm{T}}(k)\end{bmatrix}^{\mathrm{T}}$ and

$$A_\varepsilon\triangleq\begin{bmatrix}I+\varepsilon A_{11} & \varepsilon A_{12}\\ A_{21} & A_{22}\end{bmatrix},\ B_\varepsilon\triangleq\begin{bmatrix}\varepsilon B_1\\ B_2\end{bmatrix},$$

where $\varepsilon>0$ is a singularly perturbed parameter; $x_1(k)\in\mathbb{R}^{n_s}$ and $x_2(k)\in\mathbb{R}^{n_f}$ ($n_s+n_f=n$) are the slow and fast state vectors, respectively; $u(k)\in\mathbb{R}^m$ is the control input; $w(k)\in\mathbb{R}^m$ is an unknown external disturbance. The matrices A_{11}, A_{12}, A_{21}, A_{22}, B_1 and B_2 are known real matrices.

Besides, the system (1) satisfies the following assumptions:

- The input matrix B_ε is full column rank, that is, $\mathrm{rank}(B_\varepsilon)=m$;
- The external disturbance $w(k)$ possesses $\|w(k)\|\leq\varpi$, where $\varpi\geq 0$ is a known scalar.

Remark 1: In many electrical circuits, the continuous-time SPS has widely employed to model the multiple time-scale phenomena, which are often occurred due to some small "parasitic" circuit elements, such as the "parasitic" capacitance in the van der Pol oscillator circuit [1] and the modified Chua's circuit with mixed-mode oscillations [20]. With the development of the network-based communication, it seems naturally to discretize a continuous-time SPSs to its discrete-time counterpart for realizing the digital control/filtering. It is worth mentioning that different sampling rate will lead to different discrete-time SPS model [13]. As shown in [13], the fast-sampling SPS (1) is obtained from the practical continuous-time SPS by selecting the fast sampling rate as $T_f=\varepsilon$ and neglecting $o(\varepsilon)$ errors. One is the fast-sampling model [8], [18] and another is the slow-sampling model [31]. It is worth stressing that the stability of a circuit system is a key problem in practical applications [15], [16]. Therefore, it is valuable to study the control problem of the slow- or fast-sampling circuit models under a single-rate sample-data. This paper focuses on investigating the SMC problem of the circuit systems modeled by the fast-sampling SPS (1), which is still an open research issue.

B. RCTP

In networked systems, it is quite common that the network communication through a single channel is unreliable due mainly to the network-induced phenomena, such as packet dropouts. In order to improve the reliability of data transmission services, the RCTP is utilized between the sensors and the controller as shown in Fig. 1, where the channel 1 is the primary channel and the channels $i\in\{2,3,\ldots,N\}$ are called redundant channels [7], [30]. The packet dropouts over these channels are governed by the stochastic variables $\gamma_i(k)$ ($i=1,2,\ldots,N$), which are mutually independent Bernoulli distributed white sequences taking values of 0 or 1 as follows:

$$\mathrm{Prob}\{\gamma_i(k)=1\}=\bar{\gamma}_i,\ \text{and}\ \mathrm{Prob}\{\gamma_i(k)=0\}=1-\bar{\gamma}_i, \tag{2}$$

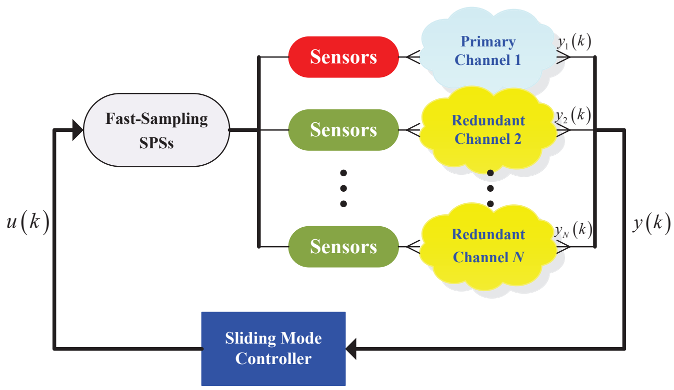

Fig. 1. Reliable SMC of fast-sampling SPSs under RCTP.

where $\bar{\gamma}_i \in [0,1]$ are known constants.

Under the above RCTP, the actual received output $y(k)$ at the controller side can be formulated as follows:

$$y(k) = \gamma_1(k)y_1(k) + \sum_{i=2}^{N}\left\{\prod_{j=1}^{i-1}(1-\gamma_j(k))\gamma_i(k)y_i(k)\right\} \quad (3)$$

where $y_i(k) \triangleq C_i x(k)$ denotes the measurement output of the ith transmission channel with the known output matrices $C_i \in \mathbb{R}^{p\times n}$. In fact, the RCTP is executed under the assumption that all the measured outputs $\{y_1(k), y_2(k), \ldots, y_N(k)\}$ are transmitted to controller simultaneously without communication time-delays. In the case that more than one measured outputs are received by the controller, the actual employed output will be determined by (3).

Remark 2: It is observed from the actual measurement model (3) that if no packet dropout occurs at the primary channel, the measured output will be $y(k) = y_1(k)$, which implies that the other redundant channels ($i = 2, 3, \ldots, N$) will not be activated. When the channels from 1 to $i-1$ suffer from packet dropouts and the ith channel transmits successfully, the measured output will be $y(k) = y_i(k)$. That is, no information would be received by the controller (i.e., $y(k) = 0$ in this case) if and only if all channels transmit unsuccessfully. Although the introduction of RCTP to the control systems will increase the cost of equipment/energy, the probability of packet dropouts is significantly reduced from $1-\bar{\gamma}_1$ to $\prod_{i=1}^{N}(1-\bar{\gamma}_i)$ and thus the network reliability is much improved via RCTP mechanism. In practical applications, the number of the redundant channels will be determined as per different engineering requirements and capacities. For example, in some cases with high requirement for the reliability, the number of the redundant channels can be determined so that the probability of packet dropouts is less than $\prod_{i=1}^{N}(1-\bar{\gamma}_i)$. However, in some cases with limited resources, one may only employ the redundant channels as more as possible.

The interest of this paper is to design a SMC law $u(k)$ such that the resultant closed-loop system is mean-square stable under the RCTP (2)–(3).

III. Design of SMC Under RCTP

A. Sliding Function and Sliding Mode Controller

It is worth stressing that in order to ensure the engineering feasibility, the sliding function and the sliding mode controller are designed just by utilizing the actual measured output signal $y(k)$ in (3), which is composed of $\{y_1(k), y_2(k), \ldots, y_N(k)\}$ under RCTP. To cope with the effect of RCTP, we introduce the following mathematical notations firstly:

$$\Upsilon(k) \triangleq \gamma_1(k)C_1 + \sum_{i=2}^{N}\left\{\prod_{j=1}^{i-1}(1-\gamma_j(k))\gamma_i(k)C_i\right\}, \quad (4)$$

$$\bar{\Upsilon} \triangleq \mathbf{E}\{\Upsilon(k)\} = \bar{\gamma}_1 C_1 + \sum_{i=2}^{N}\left\{\prod_{j=1}^{i-1}(1-\bar{\gamma}_j)\bar{\gamma}_i C_i\right\}. \quad (5)$$

Clearly, with the aid of above notations, the actual measured output under RCTP (3) becomes $y(k) = \Upsilon(k)x(k)$.

Now, based on the measurement outputs $\{y_1(k), y_2(k), \ldots, y_N(k)\}$ of the N channels, we introduce the following output-feedback sliding function:

$$\begin{aligned} s(k) =& G_\varepsilon\left\{\bar{\gamma}_1 y_1(k) + \sum_{i=2}^{N}\left\{\prod_{j=1}^{i-1}(1-\bar{\gamma}_j)\bar{\gamma}_i y_i(k)\right\}\right\} \\ =& G_\varepsilon \bar{\Upsilon} x(k), \end{aligned} \quad (6)$$

where $G_\varepsilon \in \mathbb{R}^{m\times p}$ should be satisfied:

$$G_\varepsilon \bar{\Upsilon} = B_\varepsilon^{\mathrm{T}} P_\varepsilon, \quad (7)$$

with the ε-dependent matrix $P_\varepsilon > 0$ to be determined later. Clearly, it yields that $G_\varepsilon \bar{\Upsilon} B_\varepsilon = B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon > 0$ is nonsingular.

Remark 3: To the authors' best knowledge, this paper represents the first attempt to investigate the SMC problem under the RCTP (2)–(3). A key challenge here is how to design the sliding function by just utilizing the measurement output signals $\{y_1(k), y_2(k), \ldots, y_N(k)\}$. To this end, the sliding function (6) is constructed skillfully by considering the specific communication structure of RCTP (3), which will be helpful in the stability analysis of the sliding mode dynamics subsequently. Similar to [11], if the matrix P_ε is known in the equality condition (7), the gain matrix G_ε can be obtained readily by solving the following convex optimization problem:

$$\begin{bmatrix} -\kappa I & G_\varepsilon \bar{\Upsilon} - B_\varepsilon^{\mathrm{T}} P_\varepsilon \\ \star & -I \end{bmatrix} < 0, \quad (8)$$

where $\kappa > 0$ is a specified sufficiently small scalar.

From the system (1) and the sliding function (6), we have

$$s(k+1) = G_\varepsilon \bar{\Upsilon} A_\varepsilon x(k) + G_\varepsilon \bar{\Upsilon} B_\varepsilon u(k) + D_\varepsilon(k), \quad (9)$$

where $D_\varepsilon(k) \triangleq G_\varepsilon \bar{\Upsilon} B_\varepsilon w(k)$. According to the assumption on the external disturbance $w(k)$, there exist bounds $\underline{d}_i$, $\bar{d}_i$ ($i = 1, 2, \ldots, m$) satisfying $\underline{d}_i \le d_i(k) \le \bar{d}_i$, where $d_i(k)$ are the ith element of $D_\varepsilon(k)$. Define

$$d_{io} = \frac{\bar{d}_i + \underline{d}_i}{2}, \quad D_o = \begin{bmatrix} d_{1o} & d_{2o} & \cdots & d_{mo} \end{bmatrix}^{\mathrm{T}}, \quad (10)$$

$$d_{is} = \frac{\bar{d}_i - \underline{d}_i}{2}, \quad D_s = \mathrm{diag}\{d_{1s}, d_{2s}, \ldots, d_{ms}\}. \quad (11)$$

By just utilizing the measurement outputs $\{y_i(k)\}$, we construct the following output-feedback SMC scheme:

$$u(k) = -\left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} \left[F_\varepsilon y(k) + D_o + D_s \mathrm{sgn}(s(k))\right], \quad (12)$$

where the ε-dependent gain matrix $F_\varepsilon \in \mathbb{R}^{m\times p}$ will be determined later.

Substituting (12) into (1), the following closed-loop system is yielded:

$$\begin{aligned} x(k+1) = & \left[A_\varepsilon - B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} F_\varepsilon \Upsilon(k)\right] x(k) \\ & + B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} \left[D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))\right]. \end{aligned} \quad (13)$$

In the sequel, we will analyze the stability of the closed-loop system (13) and the reachability of the sliding surface (6). To this end, the following definition and lemmas are introduced.

Definition 1: [37] The closed-loop system (13) is said to be mean-square exponentially ultimately bounded (MSEUB) if there exist constants $0 < \beta < 1$, $\alpha > 0$ and $\bar{\chi} \geq 0$ such that

$$\mathbf{E}\left\{\|x(k)\|^2\right\} \leq \alpha\beta^k \|x(0)\|^2 + \chi(k) \text{ and } \lim_{k\to\infty} \chi(k) = \bar{\chi}.$$

Lemma 1: [34] For a positive scalar $\bar{\varepsilon}$ and symmetric matrices S_1, S_2 and S_3 with appropriate dimensions, if $S_1 \geq 0$, $S_1 + \bar{\varepsilon} S_2 > 0$, $S_1 + \bar{\varepsilon} S_2 + \bar{\varepsilon}^2 S_3 > 0$ hold, then $S_1 + \varepsilon S_2 + \varepsilon^2 S_3 > 0$, $\forall \varepsilon \in (0, \bar{\varepsilon}]$ can be achieved.

Lemma 2: Defining the stochastic varying matrix $\tilde{\Upsilon}(k) \triangleq \Upsilon(k) - \bar{\Upsilon}$, then for a positive-definite matrix P and a real matrix M, it has:

$$\mathbf{E}\left\{M\tilde{\Upsilon}(k)\right\} = 0; \quad (14)$$

$$\begin{aligned} & \mathbf{E}\left\{(M\tilde{\Upsilon}(k))^{\mathrm{T}} P (M\tilde{\Upsilon}(k))\right\} \\ = & -\bar{\Upsilon}^{\mathrm{T}} M^{\mathrm{T}} P M \bar{\Upsilon} + \bar{\gamma}_1 C_1^{\mathrm{T}} M^{\mathrm{T}} P M C_1 \\ & + \sum_{i=2}^{N} \left\{\prod_{j=1}^{i-1} (1-\bar{\gamma}_j)\bar{\gamma}_i C_i^{\mathrm{T}} M^{\mathrm{T}} P M C_i\right\}. \end{aligned} \quad (15)$$

Proof: The proof is given in Appendix A. ■

Lemma 3: According to (10)–(11), it has

$$\|D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))\| \leq 2\|D_s\|. \quad (16)$$

Proof: The proof is given in Appendix B. ■

B. MSEUB of Closed-Loop System

The following theorem proposes a sufficient condition to guarantee the MSEUB of the closed-loop system (13) under the output-feedback SMC law (12) with the RCTP (2)–(3).

Theorem 1: Consider the fast sampling SPS (1) and the output-feedback SMC law (12) with the RCTP (2)–(3). For prescribed scalars $\delta > 0$, $\mu > 0$ and $\bar{\varepsilon} > 0$, if there exist symmetric matrices $P_{11} > 0$, $P_{22} > 0$, $Q > 0$, matrices P_{12} and $F_\varepsilon \in \mathbb{R}^{m\times p}$, and a scalar $\xi > 0$ such that the following LMIs hold:

$$\begin{bmatrix} -P_0 & \vec{\Xi} \\ \star & -\vec{\Lambda} \end{bmatrix} < 0, \quad (17)$$

$$\begin{bmatrix} -P_{\bar{\varepsilon}} & \hat{\Xi} \\ \star & -\hat{\Lambda} \end{bmatrix} < 0, \quad (18)$$

$$\begin{bmatrix} -P_{\bar{\varepsilon}} & \check{\Xi} \\ \star & -\check{\Lambda} \end{bmatrix} < 0, \quad (19)$$

$$\begin{bmatrix} -\xi I & \varsigma_3 I \\ \star & -B_2^{\mathrm{T}} P_{22} B_2 \end{bmatrix} < 0, \quad (20)$$

$$\begin{bmatrix} -\xi I & \varsigma_3 I \\ \star & -H \end{bmatrix} < 0, \quad (21)$$

$$\begin{bmatrix} -\xi I & \varsigma_3 I \\ \star & -\check{H} \end{bmatrix} < 0, \quad (22)$$

where $P_0 \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}$, $P_{\bar{\varepsilon}} \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} + \bar{\varepsilon} Q \end{bmatrix}$, $\varsigma_1 \triangleq \sqrt{(1+\delta)(1+\mu)}$, $\varsigma_2 \triangleq \sqrt{(1+\delta)(1+\mu^{-1}) - 1}$, $\varsigma_3 \triangleq \sqrt{1+\delta^{-1}}$, and

$$U \triangleq \begin{bmatrix} P_{11} + A_{21}^{\mathrm{T}} P_{12}^{\mathrm{T}} & P_{12} + A_{21}^{\mathrm{T}} P_{22} \\ A_{12}^{\mathrm{T}} P_{11} + A_{22}^{\mathrm{T}} P_{12}^{\mathrm{T}} & A_{12}^{\mathrm{T}} P_{12} + A_{22}^{\mathrm{T}} P_{22} \end{bmatrix},$$

$$T \triangleq \begin{bmatrix} A_{11}^{\mathrm{T}} P_{11} & A_{11}^{\mathrm{T}} P_{12} + A_{21}^{\mathrm{T}} Q \\ 0 & A_{22}^{\mathrm{T}} Q \end{bmatrix},$$

$$\begin{aligned} \vec{\Xi} \triangleq \Big[& \varsigma_1 U \quad \varsigma_2 \bar{\Upsilon}^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \quad \sqrt{\bar{\gamma}_1} C_1^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \\ & \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2} C_2^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \quad \ldots \quad \sqrt{\textstyle\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N} C_N^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \Big], \end{aligned}$$

$$\vec{\Lambda} \triangleq \mathrm{diag}\big\{P_0, B_2^{\mathrm{T}} P_{22} B_2, B_2^{\mathrm{T}} P_{22} B_2, \underbrace{B_2^{\mathrm{T}} P_{22} B_2, \cdots, B_2^{\mathrm{T}} P_{22} B_2}_{N-1}\big\},$$

$$\begin{aligned} \hat{\Xi} \triangleq \Big[& \varsigma_1 (U + \bar{\varepsilon} T) \quad \varsigma_2 \bar{\Upsilon}^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \quad \sqrt{\bar{\gamma}_1} C_1^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \\ & \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2} C_2^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \quad \ldots \quad \sqrt{\textstyle\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N} C_N^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \Big], \end{aligned}$$

$$\hat{\Lambda} \triangleq \mathrm{diag}\big\{P_{\bar{\varepsilon}}, H, H, \underbrace{H, \cdots, H}_{N-1}\big\},$$

$$H \triangleq B_2^{\mathrm{T}} P_{22} B_2 + \bar{\varepsilon}(B_2^{\mathrm{T}} P_{12}^{\mathrm{T}} B_1 + B_1^{\mathrm{T}} P_{12} B_2 + B_2^{\mathrm{T}} Q B_2),$$

$$\check{\Lambda} \triangleq \mathrm{diag}\big\{P_{\bar{\varepsilon}}, \check{H}, \check{H}, \underbrace{\check{H}, \cdots, \check{H}}_{N-1}\big\},$$

$$\begin{aligned} \check{H} \triangleq & B_2^{\mathrm{T}} P_{22} B_2 + \bar{\varepsilon}(B_2^{\mathrm{T}} P_{12}^{\mathrm{T}} B_1 + B_1^{\mathrm{T}} P_{12} B_2 + B_2^{\mathrm{T}} Q B_2) \\ & + \bar{\varepsilon}^2 B_1^{\mathrm{T}} P_{11} B_1, \end{aligned}$$

then, the closed-loop system (13) is MSEUB for any $\varepsilon \in (0, \bar{\varepsilon}]$.

Proof: We select the Lyapunov function candidate as $V(k) \triangleq x^{\mathrm{T}}(k) P_\varepsilon x(k)$. Along the closed-loop system (13), the difference of $V(k)$ is obtained by using Lemma 2 as follows:

$$\begin{aligned} & \mathbf{E}\left\{\Delta V(k) \mid x(k)\right\} \\ = & \mathbf{E}\left\{x^{\mathrm{T}}(k+1) P_\varepsilon x(k+1) \mid x(k)\right\} - x^{\mathrm{T}}(k) P_\varepsilon x(k) \\ = & x^{\mathrm{T}}(k) \Bigg\{ \left[A_\varepsilon - B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon}\right]^{\mathrm{T}} P_\varepsilon \\ & \times \left[A_\varepsilon - B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon}\right] \\ & - \bar{\Upsilon}^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} (B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon)^{-1} F_\varepsilon \bar{\Upsilon} + \bar{\gamma}_1 C_1^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} (B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon)^{-1} F_\varepsilon C_1 \\ & + \sum_{i=2}^{N} \left\{\prod_{j=1}^{i-1} (1-\bar{\gamma}_j)\bar{\gamma}_i C_i^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} (B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon)^{-1} F_\varepsilon C_i\right\} \\ & - P_\varepsilon \Bigg\} x(k) + 2x^{\mathrm{T}}(k) \left[A_\varepsilon - B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon}\right]^{\mathrm{T}} P_\varepsilon \\ & \times B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} \left[D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))\right] \\ & + \left[D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))\right]^{\mathrm{T}} \left[B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1}\right]^{\mathrm{T}} P_\varepsilon \end{aligned}$$

$$\times B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} [D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))] . \qquad (23)$$

For two adjustable scalars $\mu > 0$ and $\delta > 0$, in light of the equality condition (7), one has

$$\begin{aligned}
&2x^{\mathrm{T}}(k) \left[A_\varepsilon - B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon}\right]^{\mathrm{T}} P_\varepsilon \\
&\times B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} [D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))] \\
\leq &\delta x^{\mathrm{T}}(k) \left[A_\varepsilon - B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon}\right]^{\mathrm{T}} P_\varepsilon \\
&\times \left[A_\varepsilon - B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon}\right] x(k) \\
&+ \delta^{-1} [D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))]^{\mathrm{T}} (B_\varepsilon P_\varepsilon B_\varepsilon)^{-1} \\
&\times [D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))] , \qquad (24) \\
&\left[A_\varepsilon - B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon}\right]^{\mathrm{T}} P_\varepsilon \\
&\times \left[A_\varepsilon - B_\varepsilon \left(G_\varepsilon \bar{\Upsilon} B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon}\right] \\
\leq &(1+\mu) A_\varepsilon^{\mathrm{T}} P_\varepsilon A_\varepsilon + (1+\mu^{-1}) \bar{\Upsilon}^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \left(B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon}. \qquad (25)
\end{aligned}$$

By combining (23)–(25), it follows from (16) that for any scalar $\xi > 0$:

$$\begin{aligned}
&\mathbf{E}\left\{\Delta V(k) \mid x(k)\right\} \\
\leq &\mathbf{E}\left\{x^{\mathrm{T}}(k+1) P_\varepsilon x(k+1) \mid x(k)\right\} - x^{\mathrm{T}}(k) P_\varepsilon x(k) \\
&+ \xi \left[4\|D_s\|^2 - \|D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))\|^2\right] \\
\leq &x^{\mathrm{T}}(k) \Sigma_1(\varepsilon) x(k) + [D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))]^{\mathrm{T}} \Sigma_2(\varepsilon) \\
&\times [D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))] + 4\xi \|D_s\|^2, \qquad (26)
\end{aligned}$$

where $\Sigma_1(\varepsilon) \triangleq (1+\delta)(1+\mu) A_\varepsilon^{\mathrm{T}} P_\varepsilon A_\varepsilon + (1+\delta)(1+\mu^{-1}) \bar{\Upsilon}^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \left(B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon} - \bar{\Upsilon}^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \left(B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon\right)^{-1} F_\varepsilon \bar{\Upsilon} + \bar{\gamma}_1 C_1^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} (B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon)^{-1} F_\varepsilon C_1 + \sum_{i=2}^{N} \left\{\prod_{j=1}^{i-1} (1-\bar{\gamma}_j) \bar{\gamma}_i C_i^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} (B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon)^{-1} F_\varepsilon C_i\right\} - P_\varepsilon$ and $\Sigma_2(\varepsilon) \triangleq -\xi I + (1+\delta^{-1}) \left(B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon\right)^{-1}$.

As per the Schur complement, it is known that $\Sigma_1(\varepsilon) < 0$ is equivalent to

$$\tilde{\Sigma}_1(\varepsilon) \triangleq \begin{bmatrix} -P_\varepsilon & \Xi(\varepsilon) \\ \star & -\Lambda(\varepsilon) \end{bmatrix} < 0, \qquad (27)$$

where

$$\begin{aligned}
\Xi(\varepsilon) \triangleq \Big[\; &\varsigma_1 A_\varepsilon^{\mathrm{T}} P_\varepsilon \quad \varsigma_2 \bar{\Upsilon}^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \quad \sqrt{\bar{\gamma}_1} C_1^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \\
&\sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2} C_2^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \quad \dots \quad \sqrt{\textstyle\prod_{j=1}^{N-1} (1-\bar{\gamma}_j) \bar{\gamma}_N} C_N^{\mathrm{T}} F_\varepsilon^{\mathrm{T}} \; \Big], \\
\Lambda(\varepsilon) \triangleq \mathrm{diag}\big\{ &P_\varepsilon, B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon, B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon, \underbrace{B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon, \cdots, B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon}_{N-1} \big\},
\end{aligned}$$

and $\Sigma_2(\varepsilon) < 0$ is equivalent to

$$\tilde{\Sigma}_2(\varepsilon) \triangleq \begin{bmatrix} -\xi I & \varsigma_3 I \\ \star & -B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon \end{bmatrix} < 0. \qquad (28)$$

Let $P_\varepsilon \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} + \varepsilon Q \end{bmatrix}$. It is noted that $B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon = B_2^{\mathrm{T}} P_{22} B_2 + \varepsilon (B_2^{\mathrm{T}} P_{12}^{\mathrm{T}} B_1 + B_1^{\mathrm{T}} P_{12} B_2 + B_2^{\mathrm{T}} Q B_2) + \varepsilon^2 B_1^{\mathrm{T}} P_{11} B_1$ and $A_\varepsilon^{\mathrm{T}} P_\varepsilon = U + \varepsilon T$. With the aid of the Lemma 1, given any $\varepsilon \in (0, \bar{\varepsilon}]$, the conditions (17)–(19) ensure $\tilde{\Sigma}_1(\varepsilon) < 0$ and the conditions (20)–(22) guarantee $\tilde{\Sigma}_2(\varepsilon) < 0$, which imply that

$$\begin{aligned}
&\mathbf{E}\left\{x^{\mathrm{T}}(k+1) P_\varepsilon x(k+1) \mid x(k)\right\} - x^{\mathrm{T}}(k) P_\varepsilon x(k) \\
\leq &x^{\mathrm{T}}(k) \Sigma_1(\varepsilon) x(k) + 4\xi \|D_s\|^2 \\
\leq &-\lambda_{\min}(-\Sigma_1(\varepsilon)) \|x(k)\|^2 + 4\xi \|D_s\|^2 \\
\leq &-\psi x^{\mathrm{T}}(k) P_\varepsilon x(k) + 4\xi \|D_s\|^2, \qquad (29)
\end{aligned}$$

where $\psi \triangleq \frac{\lambda_{\min}(-\Sigma_1(\varepsilon))}{\lambda_{\max}(P_\varepsilon)} < 1$.

Then, it is obtained that

$$\begin{aligned}
&\mathbf{E}\left\{x^{\mathrm{T}}(k) P_\varepsilon x(k) \mid x(k-1)\right\} \\
\leq &(1-\psi) x^{\mathrm{T}}(k-1) P_\varepsilon x(k-1) + 4\xi \|D_s\|^2 \\
\leq &(1-\psi)^k x^{\mathrm{T}}(0) P_\varepsilon x(0) + 4\xi \|D_s\|^2 \frac{1-(1-\psi)^k}{\psi},
\end{aligned}$$

and by taking mathematical expectation again, it further leads to

$$\begin{aligned}
\mathbf{E}\left\{\|x(k)\|^2\right\} \leq &\frac{\lambda_{\max}(P_\varepsilon)}{\lambda_{\min}(P_\varepsilon)} (1-\psi)^k \|x(0)\|^2 \\
&+ 4\xi \|D_s\|^2 \frac{1-(1-\psi)^k}{\psi \lambda_{\min}(P_\varepsilon)}. \qquad (30)
\end{aligned}$$

This means that for any singular perturbed parameter $\varepsilon \in (0, \bar{\varepsilon}]$, the conditions (17)–(22) ensure that the closed-loop system (13) is MSEUB with the ultimate bound:

$$\bar{\chi} = \lim_{k\to\infty} \left\{4\xi \|D_s\|^2 \frac{1-(1-\psi)^k}{\psi \lambda_{\min}(P_\varepsilon)}\right\} = \frac{4\xi \|D_s\|^2}{\psi \lambda_{\min}(P_\varepsilon)}.$$

This completes the proof. ■

C. The Reachability of Sliding Surface

This subsection carries out the reachability analysis for the specified sliding surface $s(k) = 0$ under the output-feedback SMC law (12). From the sliding function (6) and the closed-loop system (13), we have

$$\begin{aligned}
s(k+1) = &\left[B_\varepsilon^{\mathrm{T}} P_\varepsilon A_\varepsilon - F_\varepsilon \Upsilon(k)\right] x(k) \\
&+ [D_\varepsilon(k) - D_o - D_s \mathrm{sgn}(s(k))] . \qquad (31)
\end{aligned}$$

The following theorem establishes a sufficient condition to the reachability of the specified sliding surface (6).

Theorem 2: Consider the fast sampling SPS (1) and the output-feedback SMC law (12) with the RCTP (2)–(3). For prescribed scalars $\delta > 0$, $\mu > 0$ and $\bar{\varepsilon} > 0$, if there exist symmetric matrices $P_{11} > 0$, $P_{22} > 0$, $Q > 0$, $W > 0$, matrices P_{12} and $F_\varepsilon \in \mathbb{R}^{m\times p}$, and a scalar $\xi > 0$ satisfying the following LMIs:

$$\begin{bmatrix} \bar{\Theta} & \bar{\Psi} \\ \star & \bar{\Gamma} \end{bmatrix} < 0, \qquad (32)$$

$$\begin{bmatrix} \tilde{\Theta} & \tilde{\Psi} \\ \star & \tilde{\Gamma} \end{bmatrix} < 0, \qquad (33)$$

$$\begin{bmatrix} \tilde{\Theta} & \breve{\Psi} \\ \star & \breve{\Gamma} \end{bmatrix} < 0, \qquad (34)$$

where

$$L \triangleq \begin{bmatrix} B_2^{\mathrm{T}} P_{12}^{\mathrm{T}} + B_2^{\mathrm{T}} P_{22} A_{21} & B_2^{\mathrm{T}} P_{22} A_{22} \end{bmatrix},$$

$$
\begin{aligned}
&R \triangleq \Big[\ B_1^{\mathrm{T}}P_{11}+B_2^{\mathrm{T}}P_{12}^{\mathrm{T}}A_{11}+B_1^{\mathrm{T}}P_{12}+B_2^{\mathrm{T}}QA_{21} \\
&\quad B_2^{\mathrm{T}}P_{12}^{\mathrm{T}}A_{12}+B_1^{\mathrm{T}}P_{12}A_{22}+B_2^{\mathrm{T}}QA_{22}\ \Big], \\
&J \triangleq \Big[\ B_1^{\mathrm{T}}P_{11}A_{11} \quad B_1^{\mathrm{T}}P_{11}A_{12}\ \Big], \\
&\bar{\Theta} \triangleq -\mathrm{diag}\left\{P_0, \xi I\right\}, \\
&\bar{\Gamma} \triangleq -\mathrm{diag}\big\{W, W, \underbrace{W, \ldots, W}_{N-1}, P_0, B_2^{\mathrm{T}}P_{22}B_2, B_2^{\mathrm{T}}P_{22}B_2, \\
&\underbrace{B_2^{\mathrm{T}}P_{22}B_2, \ldots, B_2^{\mathrm{T}}P_{22}B_2}_{N-1}, B_2^{\mathrm{T}}P_{22}B_2\big\}, \\
&\bar{\Psi} \triangleq \Bigg[\begin{matrix} [L-F_\varepsilon\bar{\Upsilon}]^{\mathrm{T}} & \sqrt{\bar{\gamma}_1}C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ I & 0 \end{matrix} \\
&\quad \begin{matrix} \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2}C_2^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \cdots & \sqrt{\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N}C_N^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ 0 & \cdots & 0 \end{matrix} \\
&\quad \begin{matrix} \varsigma_1 U & \varsigma_2\bar{\Upsilon}^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \sqrt{\bar{\gamma}_1}C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2}C_2^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ 0 & 0 & 0 & 0 \end{matrix} \\
&\quad \begin{matrix} \cdots & \sqrt{\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N}C_N^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & 0 \\ \cdots & 0 & \varsigma_3 I \end{matrix}\Bigg], \\
&\tilde{\Theta} \triangleq -\mathrm{diag}\left\{P_{\bar{\varepsilon}}, \xi I\right\}, \\
&\tilde{\Gamma} \triangleq -\mathrm{diag}\big\{W, W, \underbrace{W, \ldots, W}_{N-1}, P_{\bar{\varepsilon}}, H, H, \underbrace{H, \ldots, H}_{N-1}, H\big\}, \\
&\tilde{\Psi} \triangleq \Bigg[\begin{matrix} [L+\bar{\varepsilon}R-F_\varepsilon\bar{\Upsilon}]^{\mathrm{T}} & \sqrt{\bar{\gamma}_1}C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ I & 0 \end{matrix} \\
&\quad \begin{matrix} \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2}C_2^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \cdots & \sqrt{\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N}C_N^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ 0 & \cdots & 0 \end{matrix} \\
&\quad \begin{matrix} \varsigma_1 (U+\bar{\varepsilon}T) & \varsigma_2\bar{\Upsilon}^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \sqrt{\bar{\gamma}_1}C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2}C_2^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ 0 & 0 & 0 & 0 \end{matrix} \\
&\quad \begin{matrix} \cdots & \sqrt{\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N}C_N^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & 0 \\ \cdots & 0 & \varsigma_3 I \end{matrix}\Bigg], \\
&\breve{\Gamma} \triangleq -\mathrm{diag}\big\{W, W, \underbrace{W, \ldots, W}_{N-1}, P_{\bar{\varepsilon}}, \breve{H}, \breve{H}, \underbrace{\breve{H}, \ldots, \breve{H}}_{N-1}, \breve{H}\big\}, \\
&\breve{\Psi} \triangleq \Bigg[\begin{matrix} [L+\bar{\varepsilon}R+\bar{\varepsilon}^2J-F_\varepsilon\bar{\Upsilon}]^{\mathrm{T}} & \sqrt{\bar{\gamma}_1}C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ I & 0 \end{matrix} \\
&\quad \begin{matrix} \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2}C_2^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \cdots & \sqrt{\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N}C_N^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ 0 & \cdots & 0 \end{matrix} \\
&\quad \begin{matrix} \varsigma_1 (U+\bar{\varepsilon}T) & \varsigma_2\bar{\Upsilon}^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \sqrt{\bar{\gamma}_1}C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2}C_2^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ 0 & 0 & 0 & 0 \end{matrix} \\
&\quad \begin{matrix} \cdots & \sqrt{\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N}C_N^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & 0 \\ \cdots & 0 & \varsigma_3 I \end{matrix}\Bigg],
\end{aligned}
$$

and the other matrices are defined in Theorem 1, then for the singularly perturbed parameter $\varepsilon \in (0, \bar{\varepsilon}]$, the sliding variable $s(k)$ will be driven into the following sliding region $\mathbf{O}$ in mean-square sense by the output-feedback SMC law (12):

$$
\mathbf{O} \triangleq \left\{s(k) \;\middle|\; \|s(k)\| \leq \bar{\xi}\right\}, \tag{35}
$$

where $\bar{\xi} \triangleq \sqrt{\frac{4\xi\|D_s\|^2}{\lambda_{\min}(W^{-1})}}$ is the bound of the sliding region.

Proof: Consider the following Lyapunov function candidate:

$$
\tilde{V}(k) \triangleq x^{\mathrm{T}}(k)P_\varepsilon x(k) + s^{\mathrm{T}}(k)W^{-1}s(k).
$$

By utilizing the relationship (26) and the solution of (31), we evaluate the difference of $\tilde{V}(k)$ as follows:

$$
\begin{aligned}
&\mathbf{E}\left\{\Delta\tilde{V}(k) \mid x(k)\right\} \\
=&\mathbf{E}\left\{x^{\mathrm{T}}(k+1)P_\varepsilon x(k+1) \mid x(k)\right\} - x^{\mathrm{T}}(k)P_\varepsilon x(k) \\
&+ s^{\mathrm{T}}(k+1)W^{-1}s(k+1) - s^{\mathrm{T}}(k)W^{-1}s(k) \\
\leq& x^{\mathrm{T}}(k)\Sigma_1(\varepsilon)x(k) + \left[D_\varepsilon(k) - D_o - D_s\mathrm{sgn}(s(k))\right]^{\mathrm{T}}\Sigma_2(\varepsilon) \\
&\times \left[D_\varepsilon(k) - D_o - D_s\mathrm{sgn}(s(k))\right] + 4\xi\|D_s\|^2 \\
&+ x^{\mathrm{T}}(k)\Bigg\{\left[B_\varepsilon^{\mathrm{T}}P_\varepsilon A_\varepsilon - F_\varepsilon\bar{\Upsilon}\right]^{\mathrm{T}}W^{-1}\left[B_\varepsilon^{\mathrm{T}}P_\varepsilon A_\varepsilon - F_\varepsilon\bar{\Upsilon}\right] \\
&- \bar{\Upsilon}^{\mathrm{T}}F_\varepsilon^{\mathrm{T}}W^{-1}F_\varepsilon\bar{\Upsilon} + \bar{\gamma}_1C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}}W^{-1}F_\varepsilon C_1 \\
&+ \sum_{i=2}^{N}\left\{\prod_{j=1}^{i-1}(1-\bar{\gamma}_j)\bar{\gamma}_iC_i^{\mathrm{T}}F_\varepsilon^{\mathrm{T}}W^{-1}F_\varepsilon C_i\right\}\Bigg\}x(k) \\
&+ 2x^{\mathrm{T}}(k)\left[B_\varepsilon^{\mathrm{T}}P_\varepsilon A_\varepsilon - F_\varepsilon\bar{\Upsilon}\right]^{\mathrm{T}}W^{-1} \\
&\times \left[D_\varepsilon(k) - D_o - D_s\mathrm{sgn}(s(k))\right] \\
&+ \left[D_\varepsilon(k) - D_o - D_s\mathrm{sgn}(s(k))\right]^{\mathrm{T}}W^{-1} \\
&\times \left[D_\varepsilon(k) - D_o - D_s\mathrm{sgn}(s(k))\right] - s^{\mathrm{T}}(k)W^{-1}s(k) \\
\leq& \eta^{\mathrm{T}}(k)\acute{\Sigma}(\varepsilon)\eta(k) - \left[\lambda_{\min}(W^{-1})\|s(k)\|^2 - 4\xi\|D_s\|^2\right],
\end{aligned} \tag{36}
$$

where $\eta(k) \triangleq \Big[\ x^{\mathrm{T}}(k) \quad [D_\varepsilon(k) - D_o - D_s\mathrm{sgn}(s(k))]^{\mathrm{T}}\ \Big]^{\mathrm{T}}$, and

$$
\begin{aligned}
&\acute{\Sigma}(\varepsilon) \triangleq \begin{bmatrix} \acute{\Sigma}_1(\varepsilon) & \left[B_\varepsilon^{\mathrm{T}}P_\varepsilon A_\varepsilon - F_\varepsilon\bar{\Upsilon}\right]^{\mathrm{T}}W^{-1} \\ \star & \Sigma_2(\varepsilon) + W^{-1} \end{bmatrix}, \\
&\acute{\Sigma}_1(\varepsilon) \triangleq \Sigma_1(\varepsilon) + \left[B_\varepsilon^{\mathrm{T}}P_\varepsilon A_\varepsilon - F_\varepsilon\bar{\Upsilon}\right]^{\mathrm{T}}W^{-1}\left[B_\varepsilon^{\mathrm{T}}P_\varepsilon A_\varepsilon - F_\varepsilon\bar{\Upsilon}\right] \\
&\qquad + \bar{\gamma}_1C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}}W^{-1}F_\varepsilon C_1 \\
&\qquad + \sum_{i=2}^{N}\left\{\prod_{j=1}^{i-1}(1-\bar{\gamma}_j)\bar{\gamma}_iC_i^{\mathrm{T}}F_\varepsilon^{\mathrm{T}}W^{-1}F_\varepsilon C_i\right\}.
\end{aligned}
$$

By resorting to the Schur complement, it is shown that $\acute{\Sigma}(\varepsilon) < 0$ is equivalent to the following inequality hold:

$$
\grave{\Sigma}(\varepsilon) \triangleq \begin{bmatrix} \Theta(\varepsilon) & \Psi(\varepsilon) \\ \star & \Gamma(\varepsilon) \end{bmatrix} < 0, \tag{37}
$$

where

$$
\begin{aligned}
&\Theta(\varepsilon) \triangleq -\mathrm{diag}\left\{P_\varepsilon, \xi I\right\}, \\
&\Gamma(\varepsilon) \triangleq -\mathrm{diag}\big\{W, W, \underbrace{W, \ldots, W}_{N-1}, P_\varepsilon, B_\varepsilon^{\mathrm{T}}P_\varepsilon B_\varepsilon, B_\varepsilon^{\mathrm{T}}P_\varepsilon B_\varepsilon, \\
&\underbrace{B_\varepsilon^{\mathrm{T}}P_\varepsilon B_\varepsilon, \ldots, B_\varepsilon^{\mathrm{T}}P_\varepsilon B_\varepsilon}_{N-1}, B_\varepsilon^{\mathrm{T}}P_\varepsilon B_\varepsilon\big\}, \\
&\Psi(\varepsilon) \triangleq \Bigg[\begin{matrix} [B_\varepsilon^{\mathrm{T}}P_\varepsilon A_\varepsilon - F_\varepsilon\bar{\Upsilon}]^{\mathrm{T}} & \sqrt{\bar{\gamma}_1}C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ I & 0 \end{matrix} \\
&\quad \begin{matrix} \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2}C_2^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \cdots & \sqrt{\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N}C_N^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ 0 & \cdots & 0 \end{matrix} \\
&\quad \begin{matrix} \varsigma_1 A_\varepsilon^{\mathrm{T}}P_\varepsilon & \varsigma_2\bar{\Upsilon}^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \sqrt{\bar{\gamma}_1}C_1^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} \\ 0 & 0 & 0 \end{matrix} \\
&\quad \begin{matrix} \sqrt{(1-\bar{\gamma}_1)\bar{\gamma}_2}C_2^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & \cdots & \sqrt{\prod_{j=1}^{N-1}(1-\bar{\gamma}_j)\bar{\gamma}_N}C_N^{\mathrm{T}}F_\varepsilon^{\mathrm{T}} & 0 \\ 0 & \cdots & 0 & \varsigma_3 I \end{matrix}\Bigg].
\end{aligned}
$$

Let $P_\varepsilon \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22}+\varepsilon Q \end{bmatrix}$. We have the following relationships:

$$\begin{aligned} B_\varepsilon^{\mathrm{T}} P_\varepsilon B_\varepsilon =& B_2^{\mathrm{T}} P_{22} B_2 + \varepsilon(B_2^{\mathrm{T}} P_{12}^{\mathrm{T}} B_1 + B_1^{\mathrm{T}} P_{12} B_2 \\ &+ B_2^{\mathrm{T}} Q B_2) + \varepsilon^2 B_1^{\mathrm{T}} P_{11} B_1, \\ A_\varepsilon^{\mathrm{T}} P_\varepsilon =& U + \varepsilon T, \\ B_\varepsilon^{\mathrm{T}} P_\varepsilon A_\varepsilon =& L + \varepsilon R + \varepsilon^2 J. \end{aligned}$$

Thus, according to Lemma 1, it is obtained that the conditions (32)–(34) guarantee $\hat{\Sigma}(\varepsilon) < 0$ for any $\varepsilon \in (0, \bar{\varepsilon}]$, which implies that when the state trajectories escape from the region $\mathbf{O}$ around the specified sliding surface (6) (i.e., $\|s(k)\| \geq \sqrt{\frac{4\xi\|D_s\|^2}{\lambda_{\min}(W^{-1})}}$), it renders $\mathbf{E}\left\{\Delta\tilde{V}(k)\right\} < 0$ in (36). Therefore, the sliding variable $s(k)$ is strictly deceasing outside the region $\mathbf{O}$, and the reachability of the sliding region $\mathbf{O}$ is ensured in mean-square sense. ∎

It is seen from Theorems 1 and 2 that under the output-feedback SMC law (12) with the RCTP (2)–(3), the state trajectories of the closed-loop system (13) will not only be driven into the sliding region $\mathbf{O}$ around the sliding surface (6) in a finite time but also be guaranteed as MSEUB over the sliding region $\mathbf{O}$.

D. Solving Algorithm With $\bar{\varepsilon}$ Estimation

In practical applications, it is very meaningful to find the upper-bound $\bar{\varepsilon}$ for the singular perturbation parameter, which reflects the conservativeness of the proposed SMC strategy in stabilizing fast-sampling SPSs (1). Based on Theorems 1 and 2, we develop an algorithm to design the sliding surface (6) and the SMC law (12) with estimating the ε-bound as follows.

- ***Step 1.*** Choose a sufficiently small scalar $\Delta\varepsilon$, and set the scalars $\bar{\varepsilon} = \Delta\varepsilon$ and $t = 1$.
- ***Step 2.*** Let $\delta = \frac{\zeta}{1-\zeta}$ and $\mu = \frac{\iota}{1-\iota}$ with $\zeta \in (0,1)$ and $\iota \in (0,1)$. Check the feasibility of the LMIs in Theorems 1 and 2 simultaneously for prescribed adjustable scalars $\zeta = i\Delta\nu_1$ and $\iota = j\Delta\nu_2$, where $i = j = 1$, $\Delta\nu_1 > 0$ and $\Delta\nu_2 > 0$ are preassigned small fixed step sizes.
- ***Step 3.*** If the conditions have feasible solutions for $t \leq L$ (L denotes the given maximum iterative steps), set $\bar{\varepsilon} = \bar{\varepsilon} + \Delta\varepsilon$ and $t = t + 1$, go to Step 2; otherwise, modify the scalars δ and μ by adjusting i and j incrementally over $i \in \left[1, \lceil\frac{1}{\Delta\nu_1}\rceil - 1\right]$ and $j \in \left[1, \lceil\frac{1}{\Delta\nu_2}\rceil - 1\right]$ until no feasible solutions can be found, go to Step 4.
- ***Step 4.*** Output the maximum feasible bound $\bar{\varepsilon}$, and get the matrices F_ε, P_{11}, P_{12}, P_{22} and Q. For a known singular perturbation parameter $\varepsilon \in (0, \bar{\varepsilon}]$, solve the LMI (8) with a sufficiently small scalar $\kappa > 0$ and the matrix $P_\varepsilon = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22}+\varepsilon Q \end{bmatrix}$, and thus obtain the gain matrix G_ε in sliding function (6).
- ***Step 5.*** Produce the output-feedback SMC law (12) with matrices F_ε, G_ε, D_o and D_s, and apply it to the fast-sampling SPSs (1) with the singularly perturbed parameter $\varepsilon \in (0, \bar{\varepsilon}]$ and the RCTP (2)–(3).

Remark 4: For the first attempt, this paper investigates the SMC problem for the fast-sampling SPS (1) under the RCTP (2)–(3). The proposed output-feedback SMC scheme exhibits the following distinct features: 1) In order to ensure the engineering implementation of the SMC, the sliding function (6) is designed by considering the structural characteristics of the measurement output model (3) under the RCTP. 2) The established sufficient conditions for guaranteeing the MSEUB and the reachability of the SMC system are dependent on the packet loss probabilities in the RCTP (2)–(3) and the available upper bound of the singularly perturbed parameter ε. If the upper bound ϖ of the external disturbance $w(k)$ is unknown, then the stabilization problem for the fast-sampling SPS (1) under the RCTP (2)–(3) still can be solved readily by the proposed SMC scheme with combining the adaptive technique as in [6].

IV. Simulation Studies: Remote SMC of an Operational Amplifier Circuit Under the RCTP

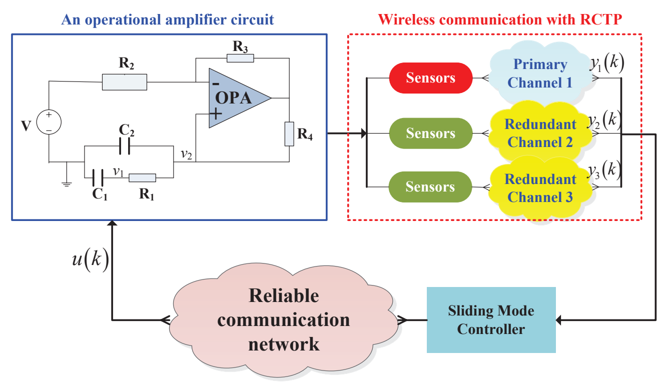

Fig. 2. Remote SMC of an operational amplifier circuit under the RCTP.

As shown in Fig. 2, we conduct a simulation for remote SMC of an operational amplifier (OPA) circuit under the RCTP. Owing to the virtual short circuit between OPA input terminals, the voltage at the inverting terminal will be equal to v_2. Thus, the current through R_2 will be $\frac{u-v_2}{R_2}$. Meanwhile, owing to the infinite input impedance of the OPA, the current through R_3 will be also $\frac{u-v_2}{R_2}$. Furthermore, it is concluded that the current through R_4 will be $\frac{(v_2-u)R_3}{R_2R_4}$. Now, by applying the Kirchoff current law to v_1 and v_2, we obtain the following state equations:

$$\dot{v}_1 = -\frac{1}{R_1\mathbf{C}_1}v_1 + \frac{1}{R_1\mathbf{C}_1}v_2, \tag{38}$$

$$\mathbf{C}_2\dot{v}_2 = \frac{1}{R_1}v_1 + \left(\frac{R_3}{R_2R_4} - \frac{1}{R_1}\right)v_2 - \frac{R_3}{R_2R_4}u. \tag{39}$$

Let $x_1(t) = v_1$ and $x_2(t) = v_2$. Similar to [1], it is supposed that $\mathbf{C}_2$ is a small “parasitic” capacitor, which can be regarded as a singularly perturbed parameter, i.e., $\mathbf{C}_2 = \varepsilon$. Other parameters are given as $R_1 = R_4 = 2\ \Omega$, $R_2 = 3\ \Omega$, $R_3 = 1\ \Omega$, $\mathbf{C}_1 = 0.3$ F. By using the discretizing approach in [13], we obtain the following discrete-time fast-sampling SPS:

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} =& \begin{bmatrix} 1-1.2927\varepsilon & 0.9921\varepsilon \\ 0.4252 & 0.7165 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ &+ \begin{bmatrix} -0.1247\varepsilon \\ -0.1417 \end{bmatrix} (u(k)+w(k)), \end{aligned} \tag{40}$$

TABLE I
RCTP ($N = 3$) IN THE EXAMPLE

Protocol Settings	Measurement matrix C_i	Probability $\bar{\gamma}_i$
Primary channel ($i = 1$)	$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\bar{\gamma}_1 = 0.7$
Redundant channel(a) ($i = 2$)	$C_2 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}$	$\bar{\gamma}_2 = 0.6$
Redundant channel(b) ($i = 3$)	$C_3 = \begin{bmatrix} 0.8 & 0.1 \\ 0 & 0.7 \end{bmatrix}$	$\bar{\gamma}_3 = 0.5$

TABLE II
FOUR DIFFERENT RCTP CASES

Cases & Solutions	$\bar{\varepsilon}$	Feasible solutions
Case 1: PC	0.3401	$F_\varepsilon = \begin{bmatrix} 0.3304 & -0.5235 \end{bmatrix}$, $P_\varepsilon = \begin{bmatrix} 11.3105 & -14.4738 \\ -14.4738 & 21.1932 + 17.8084\varepsilon \end{bmatrix}$ $W = 11.1571$, $\xi = 2.9725$
Case 2: PC+RC(a)	0.8194	$F_\varepsilon = \begin{bmatrix} 0.1017 & -0.1578 \end{bmatrix}$, $P_\varepsilon = \begin{bmatrix} 5.8083 & -4.7922 \\ -4.7922 & 9.8067 + 2.9061\varepsilon \end{bmatrix}$ $W = 11.2992$, $\xi = 2.9048$
Case 3: PC+RC(b)	0.5436	$F_\varepsilon = \begin{bmatrix} 0.2677 & -0.4166 \end{bmatrix}$, $P_\varepsilon = \begin{bmatrix} 11.1213 & -13.1923 \\ -13.1923 & 20.6578 + 5.7722\varepsilon \end{bmatrix}$, $W = 11.2425$, $\xi = 2.8263$
Case 4: PC+RC(a)+RC(b)	0.9079	$F_\varepsilon = \begin{bmatrix} 0.0757 & -0.1223 \end{bmatrix}$, $P_\varepsilon = \begin{bmatrix} 5.0070 & -3.5575 \\ -3.5575 & 8.3842 + 2.8645\varepsilon \end{bmatrix}$, $W = 11.1636$, $\xi = 2.9724$

where the matched disturbance is $w(k) = \varpi \cos(20k)$ with $\varpi = 0.1$.

The simulation experiment is conducted by MATLAB (R2014a) for the SMC of the fast-sampling SPS under the RCTR (2)–(3) with three channels (i.e., one primary channel and two redundant channels) as shown in Fig. 2, where the implementation of the OPA circuit is based on the state equation (40) by setting the fast sampling interval $T_f = \varepsilon$ as per [13] and the communication between sensors and controller is executed via IEC 62439-3-based industrial WiFi network [7]. The number of redundant channels is $N = 3$ with the measurement matrices and the probabilities of successfully transmitted packets given as in Table I.

Now, we further explore the SMC performance under different RCTP cases. By applying the algorithm in Section III-D, four cases are considered as shown in Table II, from which it is observed that for Case 4, i.e., the primary channel (PC) and the two redundant channels (RC(a) and RC(b)) are exploited simultaneously, the largest upper bound of the singular perturbation parameter ε can be obtained, while for Case 1, i.e., only PC is utilized, the smallest $\bar{\varepsilon}$ just be ensured.

In the simulation, we choose the singular perturbation parameter $\varepsilon = 0.3$ and the bounds in (11) are $\bar{d}_i = -\underline{d}_i = \varpi \|G_\varepsilon \bar{\Upsilon} B_\varepsilon\|$, that is, $D_o = 0$ and $D_s = \varpi \|G_\varepsilon \bar{\Upsilon} B_\varepsilon\|$. Then, the following reliable SMC laws are designed for the above mentioned four cases:

- Case 1 (PC):

$$
\begin{aligned}
u(k) &= \begin{bmatrix} -1.6194 & 2.5658 \end{bmatrix} y(k) - 0.1\mathrm{sgn}(s(k)), \\
y(k) &= \gamma_1(k) C_1 x(k), \\
s(k) &= \bar{\gamma}_1 G_\varepsilon y_1(k),
\end{aligned}
$$

- Case 2 (PC+RC(a)):

$$
\begin{aligned}
u(k) &= \begin{bmatrix} -0.6931 & 1.0754 \end{bmatrix} y(k) - 0.1\mathrm{sgn}(s(k)), \\
y(k) &= \gamma_1(k) C_1 x(k) + (1 - \gamma_1(k))\gamma_2(k) C_2 x(k), \\
s(k) &= G_\varepsilon \left[\bar{\gamma}_1 y_1(k) + (1 - \bar{\gamma}_1)\bar{\gamma}_2 y_2(k)\right],
\end{aligned}
$$

- Case 3 (PC+RC(b)):

$$
\begin{aligned}
u(k) &= \begin{bmatrix} -1.4574 & 2.2680 \end{bmatrix} y(k) - 0.1\mathrm{sgn}(s(k)), \\
y(k) &= \gamma_1(k) C_1 x(k) + (1 - \gamma_1(k))\gamma_3(k) C_3 x(k), \\
s(k) &= G_\varepsilon \left[\bar{\gamma}_1 y_1(k) + (1 - \bar{\gamma}_1)\bar{\gamma}_3 y_3(k)\right],
\end{aligned}
$$

- Case 4 (PC+RC(a)+RC(b)):

$$
\begin{aligned}
u(k) &= \begin{bmatrix} -0.9563 & 1.5214 \end{bmatrix} y(k) - 0.1\mathrm{sgn}(s(k)), \\
y(k) &= \gamma_1(k) C_1 x(k) + (1 - \gamma_1(k))\gamma_2(k) C_2 x(k) \\
&\quad + (1 - \gamma_1(k))(1 - \gamma_2(k))\gamma_3(k) C_3 x(k), \\
s(k) &= G_\varepsilon \left[\bar{\gamma}_1 y_1(k) + (1 - \bar{\gamma}_1)\bar{\gamma}_2 y_2(k)\right. \\
&\quad \left. + (1 - \bar{\gamma}_1)(1 - \bar{\gamma}_2)\bar{\gamma}_3 y_3(k)\right].
\end{aligned}
$$

Under the initial condition $x(0) = \begin{bmatrix} -3 & 5 \end{bmatrix}^{\mathrm{T}}$, the simulation results shown in Figs. 3–8 are obtained from MATLAB (R2014a) by simulating OPA circuit based on the state equation (40) with the fast sampling interval $T_f = \varepsilon$, where the voltages v_1 and v_2 are captured from the simulation. In practical application, the voltages v_1 and v_2 can be measured directly via voltmeters or voltage sensors. Fig. 3 shows the fast-sampling SPS (40) with the singular perturbation parameter $\varepsilon = 0.3$ is unstable. Fig. 4 depicts the random packet dropouts in three channels. The measured outputs in four cases are plotted in Fig. 5. By using the above measurement outputs, the state trajectories $x(k)$ and the sliding variable $s(k)$ of the closed-loop system are shown in Figs. 6–7, where one can observe that in Case 4, the convergence speed of $x(k)$ is fastest and the bound $\bar{\xi}$ of the sliding region is smallest. The SMC input $u(k)$ in four cases are given in Fig. 8. All simulation results illustrate that the utilization of the redundant channels can substantially improve the output-feedback SMC performance for the fast-sampling SPSs (40).

V. CONCLUSIONS

This paper has addressed the output-feedback SMC problem of the fast-sampling singularly perturbed systems. The redundant channels transmission protocol has been employed to improve the reliability of the network transmission between the sensors and the controller. An output-feedback sliding function has been constructed based on the structure characteristic of the redundant channels transmission protocol. By resorting to some appropriate Lyapunov functions, the MSEUB of the closed-loop system and the reachability of the specified sliding surface have been analyzed with estimating the available upper bound for the singularly perturbed parameter. It has been shown from a numerical example that the more redundant channels are applied, the better output-feedback SMC performance are achieved for the fast-sampling SPSs.

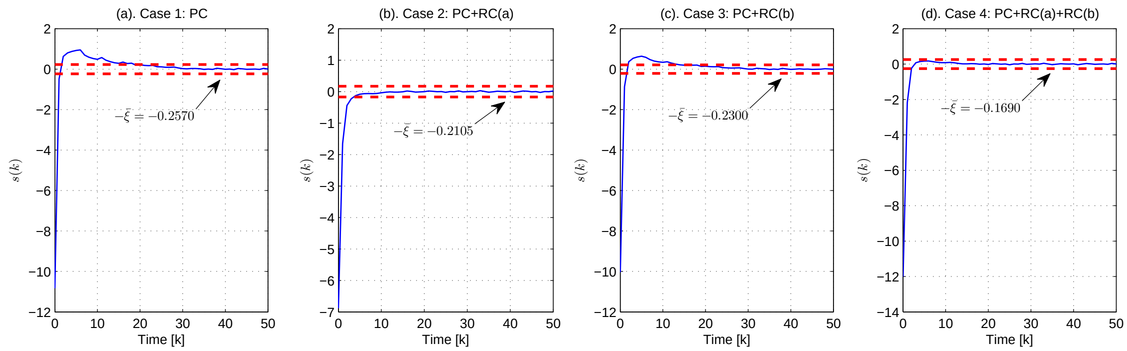

Fig. 7. Sliding variables $s(k)$ in four cases.

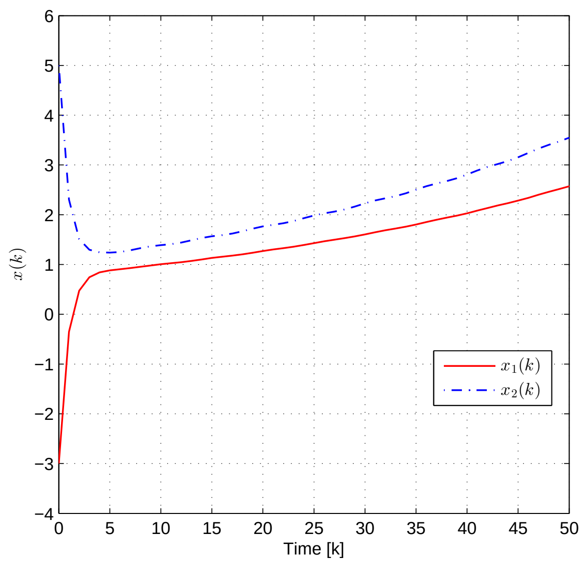

Fig. 3. State trajectories $x(k)$ in open-loop case.

Appendix

A. *Proof of Lemma 2*

The relationship in (14) can be obtained directly. Now, we prove the relationship (15). According the notations in (4)–(5), it yields

$$
\begin{aligned}
&\mathbf{E}\left\{(M\tilde{\Upsilon}(k))^{\mathrm{T}}P(M\tilde{\Upsilon}(k))\right\}\\
=&\mathbf{E}\left\{\Upsilon^{\mathrm{T}}(k)M^{\mathrm{T}}PM\Upsilon(k))\right\}-2\mathbf{E}\left\{\Upsilon^{\mathrm{T}}(k)M^{\mathrm{T}}PM\bar{\Upsilon}\right\}\\
&+\bar{\Upsilon}^{\mathrm{T}}M^{\mathrm{T}}PM\bar{\Upsilon}\\
=&-\bar{\Upsilon}^{\mathrm{T}}M^{\mathrm{T}}PM\bar{\Upsilon}+\bar{\gamma}_1C_1^{\mathrm{T}}M^{\mathrm{T}}PMC_1\\
&+\sum_{i=2}^{N}\left\{\prod_{j=1}^{i-1}(1-\bar{\gamma}_j)\bar{\gamma}_iC_i^{\mathrm{T}}M^{\mathrm{T}}PMC_i\right\}.
\end{aligned}
$$

The proof is completed.

B. *Proof of Lemma 3*

For the ith element of $D_\varepsilon(k)-D_o-D_s\mathrm{sgn}(s(k))$, we consider the following three cases:

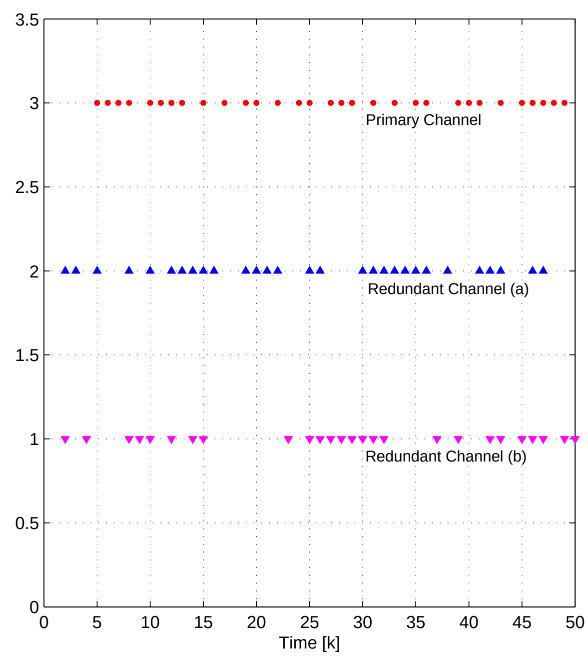

Fig. 4. Random packet dropouts in three channels.

- Case 1: $s_i(k)=0$. Then, it has

$$
\begin{aligned}
&\mid d_i(k)-d_{io}-d_{is}\mathrm{sgn}(s_i(k))\mid\\
=&\left|\frac{d_i(k)-\bar{d}_i}{2}+\frac{d_i(k)-\underline{d}_i}{2}\right|\\
\le&\left|\frac{d_i(k)-\bar{d}_i}{2}\right|+\left|\frac{d_i(k)-\underline{d}_i}{2}\right|\\
\le&2d_{is}.
\end{aligned}
$$

- Case 2: $s_i(k)>0$. Now, it gets

$$
\begin{aligned}
&\mid d_i(k)-d_{io}-d_{is}\mathrm{sgn}(s_i(k))\mid\\
=&\mid d_i(k)-\underline{d}_i\mid\\
\le&2d_{is}.
\end{aligned}
$$

- Case 3: $s_i(k)<0$. One has

$$
\begin{aligned}
&\mid d_i(k)-d_{io}-d_{is}\mathrm{sgn}(s_i(k))\mid\\
=&\mid d_i(k)-\bar{d}_i\mid
\end{aligned}
$$

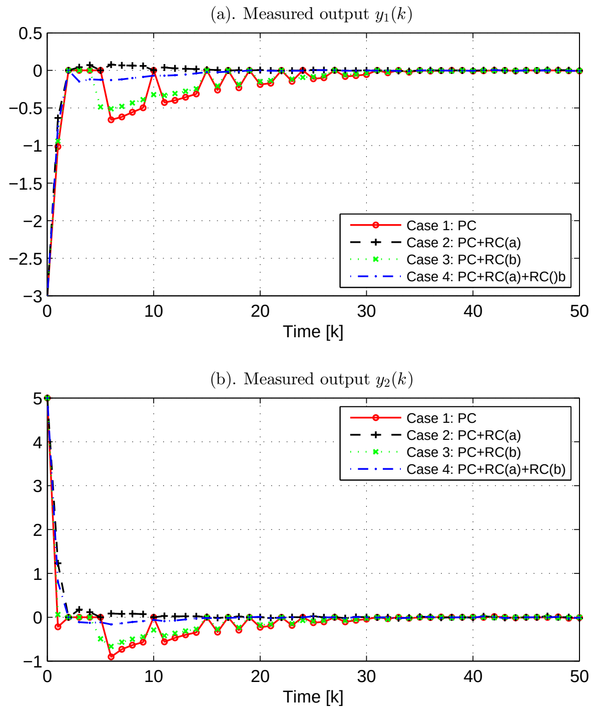

Fig. 5. Measured outputs $y(k)$ in four cases.

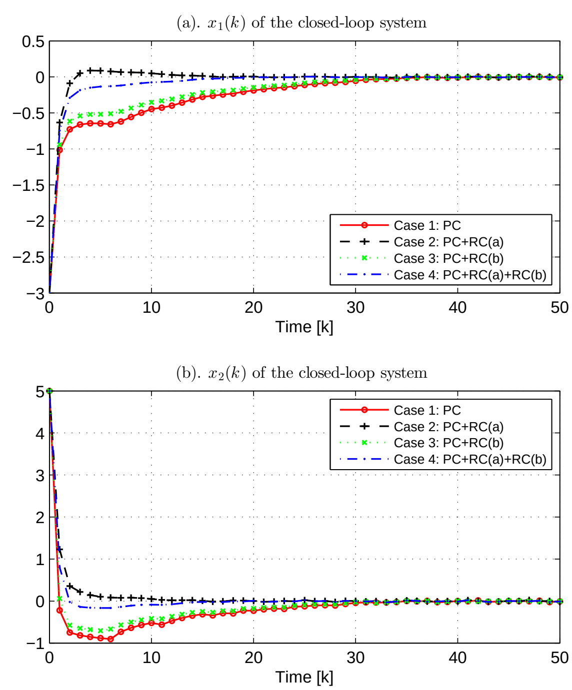

Fig. 6. State trajectories $x(k)$ in closed-loop cases.

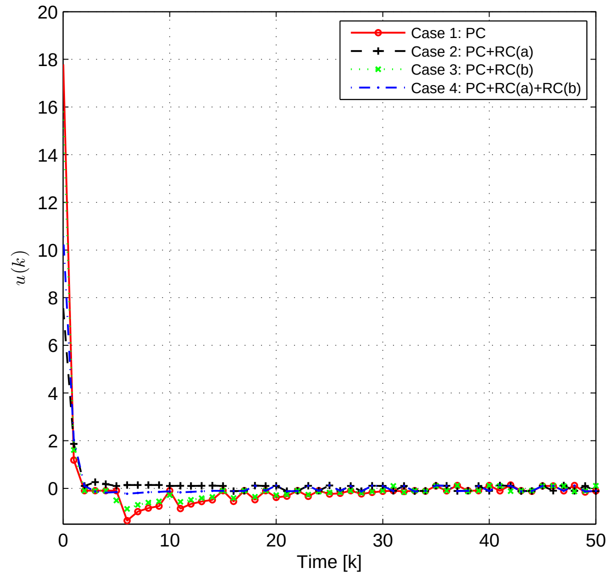

Fig. 8. SMC input $u(k)$ in four cases.

$$\leq 2d_{is}.$$

To sum up the above three cases, it always has $\mid d_i(k) - d_{io} - d_{is}\mathrm{sgn}(s_i(k)) \mid \leq 2d_{is}$. Next, by resorting to the definition of Euclidean norm, we obtain

$$\begin{aligned}
&\|D_\varepsilon(k) - D_o - D_s\mathrm{sgn}(s(k))\| \\
=&\sqrt{\sum_{i=1}^{m} |d_i(k) - d_{io} - d_{is}\mathrm{sgn}(s_i(k))|^2} \\
\leq& 2\sqrt{\sum_{i=1}^{m} d_{is}^2} = 2\|D_s\|.
\end{aligned}$$

This completes the proof.

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